... action on the
...sses of cusp inertia subgps $\subseteq \pi_1$

symmetric gp on r letters

$\to \circlearrowright \mathfrak{S}_r$

Pf of lem 1

If $r \leq 1$, then trivial

So we may assume that $r \geq 2$

To verify lemma 1,

by replacing "$X^{\log}$" (smooth log curve assoc to X) by a suitable stable log curve "$Z^{\log}$"

[cf. deformation theory and

specialization isom of log fund gp],

it suffices to show that

$\exists$ automorphis...

arbitrary ...

[$\alpha \mapsto$ ...

$Z^{\log} :=$ ...

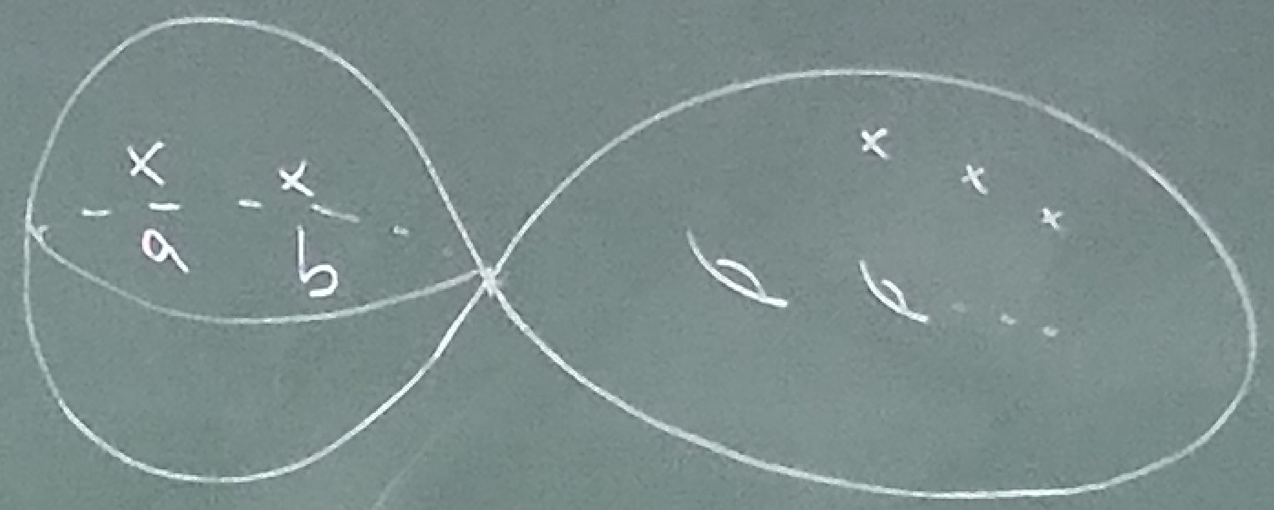
stable log curve

"$\mathcal{Z}^{\log}$"

log fund gp]

$\exists$ automorphism α of $\underline{\mathcal{Z}}^{\log}$ which induces arbitrary transposition on the set of the cusps of $\mathcal{Z}^{\log}$

$$\left[\ \alpha \rightsquigarrow \alpha_n : \mathcal{Z}_n^{\log} \xrightarrow{\sim} \mathcal{Z}_n^{\log} \rightsquigarrow \textcircled{\ } \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)!\ \right]$$

$\mathcal{Z}^{\log} :=$ 

prop3 $\rightsquigarrow$ Thm

considering the action on the set of conj classes of cusp inertia subgps $\subseteq \Pi_1$

symmetric gp on r letters

lem 1 $n \geq 1$

The composite

$$\mathrm{Out}^{FC}(\Pi_n) \longrightarrow \mathrm{Out}^{FC}(\Pi_1) \longrightarrow \mathfrak{S}_r$$

is surjective

"prop3 $\rightsquigarrow$ Thm"

follows from lem 1 //

Mochizuki

: On the Combinatorial cuspidalization of hyperbolic curves

Cor 3.4

· Prop2 $\rightsquigarrow$ Prop3 follows from

Fact $\mathrm{Out}^{FC}(\Pi_n)^{\Delta +} = \mathrm{Out}^{FC}(\Pi_n)^{\mathrm{cusp}}$ $(n \geq 4)$

by replacing

[cf. deformat

it suffices to show

prop 1 $\rightsquigarrow$ prop 2

Def $(\Delta +)$

· X : tripod

...ction on the
...s of cusp inertia subgps $\subseteq \Pi_1$

Symmetric gp on r letters

$\circlearrowright \mathfrak{S}_r$

Mochizuki
On the combinatorial
cuspidalization
hyperbolic curves

Cor 3.8

(4)

prop 1 $\rightsquigarrow$ prop 2

<u>Def</u> ($\Delta +$)

- X: tripod

Note that $X_n \cong \mathcal{M}_{0,n+3}$ moduli stack (marked pts are ordered)

$\rightsquigarrow \mathfrak{S}_{n+3} \longrightarrow \mathrm{Out}(\pi_1(\mathcal{M}_{0,n+3})) \cong \mathrm{Out}(\Pi_n)$

Sym gp

$\mathrm{Im} \ni$ outer modular symmetry

- $\mathrm{Out}^{FCS}(\Pi_n) := \left\{ \sigma \in \mathrm{Out}^{FC}(\Pi_n) \;\middle|\; \begin{array}{l} \sigma\tau = \tau\sigma \\ \text{for } \forall \tau\text{: outer modular sym} \end{array} \right\}$

$z^{\log} :=$ $-\frac{x}{a}$ $-\frac{x}{b}$

- ι : Ou...

...stack

...d pts are ordered)

(Π_n)

...f.

$\sigma\tau = \tau\sigma$

fn $\forall \tau$: outer, modular, sym $\}$

• $\iota : \mathrm{Out}^{FC}(\Pi_2) \hookrightarrow \mathrm{Out}^{FC}(\Pi_1)$ natural inj

$GT := \iota(\mathrm{Out}^{FCP}(\Pi_2))$

↳ Grothendieck-Teichmüller gp

• $X^{\log}$: smooth log curve of type (g, r): arbitrary, $n \geq 2$

$$\pi(Z_2^{\log}) \cong$$

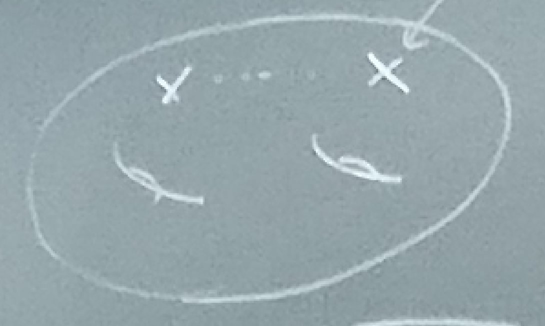
$$\begin{tikzcd}
 & \Pi_{n/n-1} \arrow[r, equal] \arrow[d] & \Pi_{n/n-1} \arrow[d] & & \\
1 \arrow[r] & \Pi_{n/n-2} \arrow[r] \arrow[d] & \Pi_n \arrow[r, two heads] \arrow[d] & \Pi_{n-2} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Pi_{n-1/n-2} \arrow[r] & \Pi_{n-1} \arrow[r, two heads] & \Pi_{n-2} \arrow[r] & 1
\end{tikzcd}$$

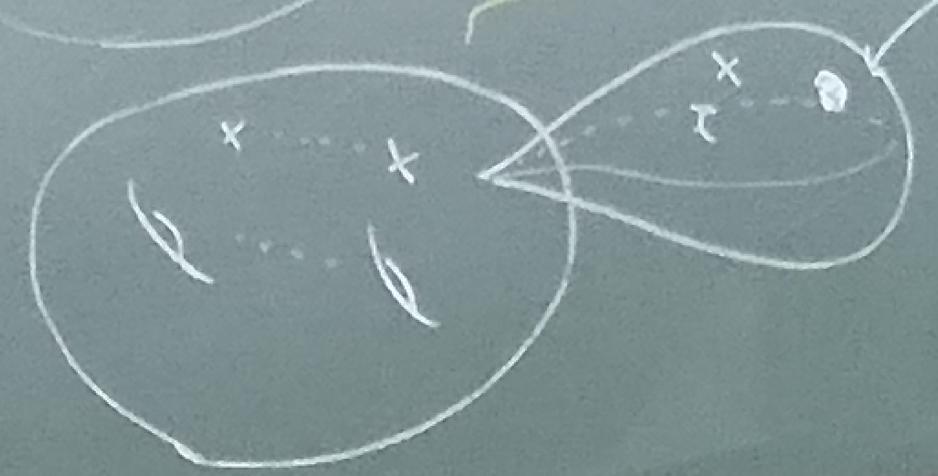
$\Pi_{n-1/n-2} \cong \pi_1(Z^{\log})$

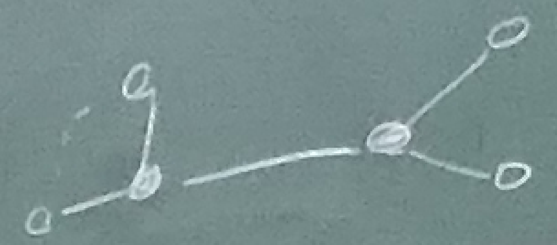
$Z^{\log}$: geom gen fiber of $X_{n-1}^{\log} \to X_{n-2}^{\log}$

For $\forall x \in Z^{\log}$: cusp

E_x : minor cuspidal component at x

e.g. $Z^{\log} =$ 

$\left(Z^{\log}\right)_x =$ 



$\left(\pi_1\left((Z^{\log}_{Z})_x \right) \cong \Pi_{n/n-1} \right)$

$\rightsquigarrow T_x : \mathrm{Out}^{FC}(\Pi_n)^{\mathrm{cusp}} \longrightarrow \mathrm{Out}^{FC}(\Pi_{E_x}) \supseteq$ "GT"

tripod homomorphism

[cf. CombGC]

$\rightsquigarrow \mathrm{Out}^{FC}(\Pi_n)^{\Delta +} := \bigcup_{x : \mathrm{cusp} \in Z^{\log}} T_x^{-1}(GT) \overset{\text{subset}}{\subseteq} \mathrm{Out}^{FC}(\Pi_n)^{\mathrm{cusp}}$

Sym gp

$\cdot\ \mathrm{Out}^{FCS}($

Rmk $\mathrm{Out}^{FCP}($

$\{$

$\boxed{\text{RmkA}}$ $\mathrm{Out}^{FCP}(\Pi_n) \subseteq \mathrm{Out}^{FC}(\Pi_n)$

$$\|$$

$$\{\sigma \in \mathrm{Out}^{FC}(\Pi_n) \mid \sigma\varepsilon = \varepsilon\sigma \quad \forall \varepsilon \in \mathrm{Im}(\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n))\}$$

In fact, $\mathrm{Out}^{FCP}(\Pi_n) = \mathrm{Out}^{FC}(\Pi_n)$

($\because$ $\mathrm{Out}^{FC}(\Pi_n) \longrightarrow \mathrm{Out}^{FC}(\Pi_1)$: inj and, the fact that "not depend proj")

$\boxed{\text{RmkB}}$ (i) $n \geq 4 \Longrightarrow$ $\tau := T_x(\sigma)$ arises from $\mathrm{Out}^{FC}(\Pi_2^{tpd}) \ni \tau_2$, $\tau_2 \mapsto \tau$

$\mathrm{Out}^{FC}(\Pi_n)^{cusp} \ni \sigma$

(ii) "FCP-sym" of τ_2 is induced from "FCP-sym" of σ

(iii) (i)+(ii) + Rmk A implies $\mathrm{Out}^{FC}(\Pi_n)^{\Delta +} = \mathrm{Out}^{FC}(\Pi_n)^{cusp}$ $(n \geq 4)$